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Duality in noncommutative topologically massive gauge field theory revisited

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Abstract. We introduce a master action in non-commutative space, out of which we obtain the action of the non-commutative Maxwell–Chern–Simons theory. Then, we look for the corresponding dual theory at both first and second order in the non-commutative parameter. At the first order, the dual theory happens to be, precisely, the action obtained from the usual commutative self-dual model by generalizing the Chern–Simons term to its non-commutative version, including a cubic term. Since this resulting theory is also equivalent to the non-commutative massive Thirring model in the large fermion mass limit, we remove, as a byproduct, the obstacles arising in the generalization to non-commutative space, and to the first non-trivial order in the non-commutative parameter, of the bosonization in three dimensions. Then, performing calculations at the second order in the non-commutative parameter, we explicitly compute a new dual theory which differs from the non-commutative self-dual model and, further, differs also from other previous results and involves a very simple expression in terms of ordinary fields. In addition, a remarkable feature of our results is that the dual theory is local, unlike what happens in the non-Abelian, but commutative case. We also conclude that the generalization to non-commutative space of bosonization in three dimensions is possible only when considering the first non-trivial corrections over ordinary space.

1 Introduction

The well-known duality between the Maxwell–Chern–Simons (MCS) theory [1] and the self-dual (SD) model [2] was established a long time ago in [3], both by comparing the corresponding equations of motion, and by introducing a master action out of which the MCS theory and the SD model can be obtained.[1] Such duality leads to two equivalent descriptions of the dynamics of a parity violating, massive, spin one field. In particular, an important application of this duality is bosonization in three dimensions [6, 7] of a theory of massive self-interacting fermions, namely, the massive Thirring (MT) model. Such a bosonization was carried out in [6] by establishing, to leading order in the inverse fermionic mass, an identity between the partition functions of the MT and SD models, and then, by making use of the equivalence between the SD model and the MCS theory. In this way, the MT model is bosonized, and even when it has no manifest local gauge invariance, the bosonized theory is indeed a manifestly gauge invariant theory, namely, the MCS theory.

In view of the relevance of the above results, it would be interesting to investigate the possibility of their extension to non-commutative (NC) space. During the last few years, the interest in NC field theories has intensified, due to applications in string theory [8–10], thus giving rise to a series of new developments and applications (see [11, 12] for reviews). In NC space, the usual product is replaced by the star product of the form

$$\hat{g}(x)\star\hat{h}(x) = \exp\left[\frac{\mathrm{i}}{2}\,\theta^{\alpha\beta}\partial^{\hat{g}}_{\alpha}\partial^{\hat{h}}_{\beta}\right]\hat{g}(x)\hat{h}(x)$$
$$= \hat{g}(x)\hat{h}(x) + \frac{\mathrm{i}}{2}\,\theta^{\alpha\beta}\partial_{\alpha}\hat{g}(x)\partial_{\beta}\hat{h}(x) + O(\theta^2)\ , \qquad (1)$$

where $\hat{g}(x)$ and $\hat{h}(x)$ are arbitrary functions, and the non-commutativity parameter $\theta^{\alpha\beta}$ is an antisymmetric constant tensor. A relation between NC and ordinary spaces is given by the Seiberg–Witten map (SWM) [10], which interpolates between a NC gauge theory and its commutative counterpart, in such a way that NC gauge orbits are mapped into ordinary ones. The SWM on a NC Abelian gauge field $\hat{A}_\mu$

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[1] See [4] for a review in the use of the master action approach in diverse areas and [5] for the application of the master action approach to the context of bosonic p-branes.

is given by

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\,\theta^{\alpha\beta} A_\alpha (2\partial_\beta A_\mu - \partial_\mu A_\beta) + O(\theta^2)\,. \quad (2)$$

In particular, the NC-CS action of the form

$$I_{\text{NC-CS}} \sim \int \mathrm{d}^3x\ \epsilon^{\alpha\mu\nu} \hat{A}_\alpha \star \left(\hat{F}_{\mu\nu} + \frac{2\mathrm{i}}{3}\,\hat{A}_\mu \star \hat{A}_\nu \right)\,, \quad (3)$$

was considered in [13], where it was shown that it is mapped, via (2), into the standard commutative CS action. In the above equation, $\hat{F}_{\mu\nu}$ is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}\left[\hat{A}_\mu\,,\ \hat{A}_\nu\right]_\star\,, \quad (4)$$

where

$$\left[\hat{A}_\mu\,,\ \hat{A}_\nu\right]_\star = \hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu\,. \quad (5)$$

Notice the presence in (3) of a cubic term, even when we are considering the Abelian case. As shown in [13], such a cubic term is cancelled by the SWM.

The extension to the NC case of the duality between MCS theory and the SD model was analyzed in [14–16] (see [17] for results in the non-Abelian case). In particular, the proposal in [14] was to consider calculations at the first non-trivial order in the NC parameter, and perform an inverse SWM to the usual master action in commutative space, as defined in [3]. This leads to a master action in NC space out of which to obtain, by following a procedure analogous to the one in [3], two theories which are in fact equivalent, and which could be considered, in principle, as the NC versions of the MCS theory and the SD model. In fact, the proposal in [14] for the NC-MCS action is just the result obtained by performing the inverse SWM on the usual action of the MCS theory.

Later, [15] considered the generalization to the NC plane of the bosonization of the MT model as performed in [6]. It was shown that the NC-MT model bosonizes, in the large fermion mass limit, into a model described, up to some conventional multiplicative coefficient, by the following action in NC space:

$$I_B = \int \mathrm{d}^3x \left[-\mu \hat{f}^\mu \star \hat{f}_\mu + \frac{\kappa}{2}\,\epsilon^{\alpha\mu\nu} \hat{f}_\alpha \star \left(\partial_\mu \hat{f}_\nu - \frac{2\mathrm{i}}{3}\,\hat{f}_\mu \star \hat{f}_\nu \right) \right]\,, \quad (6)$$

where μ and κ are constant coefficients. Since the above action differs from the version of the NC-SD model given in [14], the conclusion in [15] was that, even when the NC-MT model can be bosonized in powers of the inverse fermion mass, the duality between NC-MT model and NC-MCS theory is lost. It was stated that NC-MCS is not the dual of the NC-MT model.

On the other hand, the proposal in [16] was to consider the NC-MCS theory as defined, up to some multiplicative coefficient, by the action (whose non-Abelian version was also considered in [17])

$$I_{\text{NC-MCS}} = \int \mathrm{d}^3x \left[-\frac{\kappa^2}{2\mu}\,\hat{F}^{\mu\nu} \star \hat{F}_{\mu\nu} - \kappa\epsilon^{\alpha\mu\nu} \hat{A}_\alpha \star \left(\hat{F}_{\mu\nu} + \frac{2\mathrm{i}}{3}\,\hat{A}_\mu \star \hat{A}_\nu \right) \right]\,. \quad (7)$$

Our particular choice of the multiplicative coefficients will be clarified later. It can be verified that, in fact, the above action differs from the one obtained in [14] by performing an inverse SWM on the usual commutative action of the MCS theory. Notice, however, that (7) is also a natural choice, as it makes use of the usual expressions of the NC Maxwell and CS theories. We emphasize that, even when the NC-CS action (3) is mapped into its commutative version via the SWM, the same does not happen to the Maxwell term.

The formulation in [16], which considers calculations at the second order in the NC parameter, involves to write (7) in terms of ordinary fields, and then to define a master action out of which it can be obtained. Then, starting from such master action, [16] computed another version of the NC-SD model, which differs from the one considered in [14, 15], and involves an expression which is written in terms of commutative fields.

At this point we conclude that, in fact, there exists an ambiguity in the proper definition of the NC generalizations of the MCS theory and the SD model, and this fact was indeed recognized in [15], which wondered what theory should be referred to as the NC-SD model. One possible choice would be the one obtained by replacing the ordinary product by the star one, as in [14]. Such a version has the advantage that it obeys the self-dual equation, and is the one adopted in [15]. On the other hand, (6) is perhaps a more natural choice, as the mass term remains as such, and the CS term is generalized to its NC version (3). In addition, as we have pointed out before, the action of the NC-MCS theory considered in [14, 15] differs from the one introduced in [16], namely (7).

Summarizing, we are facing two difficulties in the generalization to the NC space of the duality between MCS theory and the SD model, namely, the existence of ambiguities when defining their corresponding NC versions, and the fact that a complete bosonization, along the lines of [6], has not been formulated yet (even when [15] managed to solve a part of the problem).

The purpose of this paper is to try to shed some light on the above detailed problems, by performing a careful analysis at a perturbative order in the NC parameter.

A strong indication on the way to do this was given in [17], dealing with non-Abelian CS theories in NC space, where it was shown, using the traditional master action approach at a perturbative level in the field but to all orders in the NC parameter, that the NC Yang–Mills–Chern–Simons action of the form

$$I_{\text{NC-YMCS}} = \int \mathrm{d}^3x\ \mathrm{Tr} \left[-\frac{\kappa^2}{2\mu}\,\hat{\mathcal{F}}^{\mu\nu} \star \hat{\mathcal{F}}_{\mu\nu} - \kappa\epsilon^{\alpha\mu\nu} \hat{\mathcal{A}}_\alpha \star \left(\hat{\mathcal{F}}_{\mu\nu} + \frac{2\mathrm{i}}{3}\,\hat{\mathcal{A}}_\mu \star \hat{\mathcal{A}}_\nu \right) \right]\,, \quad (8)$$

where $\hat{\mathcal{A}}_\mu$ is a NC field in the adjoint representation of an arbitrary non-Abelian gauge group, is dual to an action which differs from the non-Abelian NC-SD model only by terms of the fourth order in the field, namely

$$
\begin{aligned}
I &= I_{\mathrm{NC-SD}}^{\mathrm{non-Abelian}} + O\left(\hat{\mathcal{B}}^4\right) \\
&= \int \mathrm{d}^3x \; \mathrm{Tr}\Big[-\mu \hat{\mathcal{B}}^\mu \star \hat{\mathcal{B}}_\mu \\
&\quad + \frac{\kappa}{2}\,\epsilon^{\alpha\mu\nu}\hat{\mathcal{B}}_\alpha \star \left(\partial_\mu \hat{\mathcal{B}}_\nu - \frac{2\mathrm{i}}{3}\,\hat{\mathcal{B}}_\mu \star \hat{\mathcal{B}}_\nu\right)\Big] \\
&\quad + O\left(\hat{\mathcal{B}}^4\right) . \qquad (9)
\end{aligned}
$$

The master action proposed in [17] was the natural generalization of the one which is usually utilized in the commutative case [3, 6, 18] (see also [19] for a master action which has a gauge invariance in all fundamental fields).

The above considerations motivate us to wonder what the results in the NC Abelian case would be like. We suspect that, since non-commutativity resembles in some respects non-Abelian structures (notice, for example, the presence of a cubic term in the Abelian NC-CS action (3)), then it will be the case that, in the NC Abelian situation, the fourth order term in the above equation is still present. However, such a result arises when considering calculations involving all orders in the NC parameter. In principle, it could be possible that more useful results would arise when considering a perturbative approach. In that respect, we point out that, until today, most results in NC space involve corrections of the first order in θ over ordinary space, and this encourages us to perform calculations at orders $O(\theta)$ and $O(\theta^2)$ and see what our results look like.

Going on along this line of thought, our proposal here is to find, by performing calculations at a perturbative level, the dual theory of (7), written in terms of ordinary fields. In doing this, we will separate our calculations into two stages. At the first part of calculations, performed in Sect. 2, we will find that, at order $O(\theta)$, the dual theory of $I_{\mathrm{NC-MCS}}$ (as defined through (7)) is precisely (6). Notice that this allows, by using the result in [15] that (6) is obtained by bosonizing the NC-MT model in the large fermion mass limit, to remove the obstacles arising in the generalization of the bosonization in three dimensions, along the lines of [6], to the NC case. We emphasize that this result holds provided that only the first non-trivial corrections over ordinary space are considered.

Then, at the second part of our calculations, performed in Sect. 3, we will find that, at order $O(\theta^2)$, the "non-Abelian-like" nature of NC theories finally prevails, and the duality between (6) and (7) is lost.[2] However, by computing the explicit form of the dual theory, we will show a remarkable result, namely, that it is local, unlike what happens to the non-Abelian, but commutative case [21]. In addition, we will show that our dual theory differs from the one computed in [16], and involves a much simpler expression written in terms of ordinary fields. We will also discuss the form of higher order contributions.

[2] The conjecture that duality should be lost when considering higher orders of the NC generalization of the theory is also suggested by the recent result in [20], which considers a NC chiral boson action and shows that, for such model, self-duality is not maintained.

2 First order

We begin our calculations by introducing the following master action in NC space

$$
\begin{aligned}
I_{\mathrm{M}} = \int \mathrm{d}^3x \; \Big[&-\mu \hat{f}^\mu \star \hat{f}_\mu \qquad (10)\\
&+\kappa\epsilon^{\alpha\mu\nu}\left(\hat{f}_\alpha \star \hat{F}_{\mu\nu} - \hat{A}_\alpha \star \left(\hat{F}_{\mu\nu} + \frac{2\mathrm{i}}{3}\,\hat{A}_\mu \star \hat{A}_\nu\right)\right)\Big] .
\end{aligned}
$$

In order to show the duality between the actions (6) and (7), we will verify that solving I_{M}, first for $\hat{f}_\mu$ (in terms of $\hat{A}_\mu$) and then for $\hat{A}_\mu$ (in terms of $\hat{f}_\mu$), we recover both actions (7) and (6), respectively. Throughout this paper, we consider boundary conditions such as surface terms in the action vanish. We begin by focusing on the NC-MCS theory (7). From (10), we find the following equation of motion for $\hat{f}^\mu$:

$$\hat{f}^\mu = \frac{\kappa}{2\mu}\,\epsilon^{\mu\alpha\beta}\hat{F}_{\alpha\beta} \, , \qquad (11)$$

and introducing this back into (10), we get the NC-MCS action (7).

Now we focus on the NC-SD model (6). We compute the equation of motion for $\hat{A}_\mu$ in (10). By performing calculations analogous to the ones in [17], we arrive at

$$2\hat{F}_{\mu\nu} = \partial_\mu \hat{f}_\nu - \partial_\nu \hat{f}_\mu - \mathrm{i}\left[\hat{A}_\mu, \hat{f}_\nu\right]_* + \mathrm{i}\left[\hat{A}_\nu, \hat{f}_\mu\right]_* . \qquad (12)$$

We first consider calculations at the first non-trivial order in θ. In order to solve the above equation, we will write it in terms of ordinary fields (A_μ, f_μ). This is done by performing a SWM of the form (2) to the gauge field $\hat{A}_\mu$. In addition, and taking into account that, in a non-gauge theory, non-commutativity affects only products of fields in the action, without changing the fields structures, we should also set the simple identity $\hat{f}_\mu = f_\mu$ [14]. However, and just in order to be general, we will instead consider a mapping of the form

$$\hat{f}_\mu = f_\mu + \theta^{\alpha\beta} b_{\mu\alpha\beta}(f_\nu) + O(\theta^2) \, , \qquad (13)$$

where $b_{\mu\alpha\beta}(f_\nu)$ is an arbitrary function of f_ν. In particular, the natural choice $\hat{f}_\mu = f_\mu$ corresponds to the particular case $b_{\mu\alpha\beta} = 0$. We will show that, in fact, our final result (the duality, at the first non-trivial order in θ, between the actions (6) and (7)) holds for any choice of $b_{\mu\alpha\beta}$. The only assumption that we will make is that $b_{\mu\alpha\beta}$ can be expanded as

$$b_{\mu\alpha\beta}(f_\nu) = \sum_{n\geq 0} b^{(n)}_{\mu\alpha\beta}(f_\nu) \, , \qquad (14)$$

where $b^{(n)}_{\mu\alpha\beta}$ is of order $O(f^n)$.

From (1), (2), (4) and (13), we write (12) as

$$\begin{aligned} &2F_{\mu\nu} + 2\theta^{\alpha\beta}\left(F_{\mu\alpha}F_{\nu\beta} - A_\alpha\partial_\beta F_{\mu\nu}\right) \\ &= \partial_\mu f_\nu - \partial_\nu f_\mu + \theta^{\alpha\beta}\left(\partial_\alpha A_\mu \partial_\beta f_\nu + \partial_\alpha f_\mu \partial_\beta A_\nu\right) \\ &+ \theta^{\alpha\beta}\left(\partial_\mu b_{\nu\alpha\beta} - \partial_\nu b_{\mu\alpha\beta}\right) + O(\theta^2)\ , \end{aligned} \tag{15}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Now we must look for a solution $A_\mu[f_\nu]$ to the above equation. In order to do this, we follow [17, 18] and consider that the solution can be expanded as

$$A_\mu = \sum_{n\geq 0} A^{(n)}_\mu[f_\nu]\ , \tag{16}$$

where $A^{(n)}_\mu[f_\nu]$ is of order $O(f^n)$. Thus, we will solve (15) order by order in f, and also in θ, by expanding each term $A^{(n)}_\mu$ in orders of θ as follows:

$$A^{(n)}_\mu = {}^{(0)}A^{(n)}_\mu + {}^{(1)}A^{(n)}_\mu(\theta) + O(\theta^2)\ . \tag{17}$$

In this way, ${}^{(p)}A^{(n)}_\mu$ is of order n in f and of order p in θ (where $p = 0, 1$).

In solving (15), we will drop pure-gauge terms of the form ${}^{(p)}F^{(n)}_{\mu\nu} = 0$, due to the gauge invariance of (10) (notice that the formulation in terms of ordinary fields inherits, via the SWM, an ordinary gauge invariance from the NC gauge invariance of (10)).

To the lowest order in f, we get from (15)

$$\begin{aligned} &2F^{(0)}_{\mu\nu} + 2\theta^{\alpha\beta}\left(F^{(0)}_{\mu\alpha}F^{(0)}_{\nu\beta} - A^{(0)}_\alpha\partial_\beta F^{(0)}_{\mu\nu}\right) \\ &= \theta^{\alpha\beta}\left(\partial_\mu b^{(0)}_{\nu\alpha\beta} - \partial_\nu b^{(0)}_{\mu\alpha\beta}\right) + O(\theta^2)\ , \end{aligned} \tag{18}$$

and using (17) (i.e. solving order by order in θ) we find, up to some pure-gauge term

$$A^{(0)}_\mu = \frac{1}{2}\,\theta^{\alpha\beta} b^{(0)}_{\mu\alpha\beta} + O(\theta^2)\ . \tag{19}$$

Next, to order $O(f)$ (15) reads (notice that ${}^{(0)}A^{(0)}_\mu$ does not contribute)

$$2F^{(1)}_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu + \theta^{\alpha\beta}\left(\partial_\mu b^{(1)}_{\nu\alpha\beta} - \partial_\nu b^{(1)}_{\mu\alpha\beta}\right) + O(\theta^2)\ , \tag{20}$$

which has the solution (up to some pure-gauge term)

$$A^{(1)}_\mu = \frac{1}{2}\,f_\mu + \frac{1}{2}\,\theta^{\alpha\beta} b^{(1)}_{\mu\alpha\beta}(f_\nu) + O(\theta^2)\ . \tag{21}$$

Now we consider the order $O(f^2)$. From (15) we get (we emphasize that ${}^{(0)}A^{(0)}_\mu$ does not contribute)

$$\begin{aligned} &2F^{(2)}_{\mu\nu} + 2\theta^{\alpha\beta}\left(F^{(1)}_{\mu\alpha}F^{(1)}_{\nu\beta} - A^{(1)}_\alpha\partial_\beta F^{(1)}_{\mu\nu}\right) \\ &= \theta^{\alpha\beta}\left(\partial_\alpha A^{(1)}_\mu \partial_\beta f_\nu + \partial_\alpha f_\mu \partial_\beta A^{(1)}_\nu\right) \\ &+ \theta^{\alpha\beta}\left(\partial_\mu b^{(2)}_{\nu\alpha\beta} - \partial_\nu b^{(2)}_{\mu\alpha\beta}\right) + O(\theta^2)\ . \end{aligned} \tag{22}$$

Using (21), we find the following solution to the above equation (up to some pure-gauge term):

$$\begin{aligned} A^{(2)}_\mu = \theta^{\alpha\beta}\Bigg[&\frac{1}{8}\,f_\alpha\left(2\partial_\beta f_\mu - \partial_\mu f_\beta\right) \\ &+ \frac{1}{2}\,b^{(2)}_{\mu\alpha\beta}(f_\nu) + H_{\mu\alpha\beta}(f_\nu)\Bigg] + O(\theta^2)\ , \end{aligned} \tag{23}$$

where

$$\partial_\mu H_{\nu\alpha\beta} = \frac{1}{8}\,\partial_\alpha f_\mu \partial_\beta f_\nu\ . \tag{24}$$

Finally, it can be shown that the solution to order $O(f^n)$, with $n \geq 3$, is given by (up to some pure-gauge term)

$$A^{(n)}_\mu = \frac{1}{2}\,\theta^{\alpha\beta} b^{(n)}_{\mu\alpha\beta}(f_\nu) + O(\theta^2) \qquad (n \geq 3). \tag{25}$$

Summarizing, from (14), (19), (21), 23) and (25) we get the following solution to (15):

$$\begin{aligned} A_\mu = \frac{1}{2}\,f_\mu + \theta^{\alpha\beta}\Bigg[&\frac{1}{8}\,f_\alpha\left(2\partial_\beta f_\mu - \partial_\mu f_\beta\right) \\ &+ \frac{1}{2}\,b_{\mu\alpha\beta}(f_\nu) + H_{\mu\alpha\beta}(f_\nu)\Bigg] + O(\theta^2)\ , \end{aligned} \tag{26}$$

where $H_{\mu\alpha\beta}(f_\nu)$ satisfies (24). From (2) and (26) we find

$$\hat{A}_\mu = \frac{1}{2}\,f_\mu + \frac{1}{2}\,\theta^{\alpha\beta} b_{\mu\alpha\beta}(f_\nu) + \theta^{\alpha\beta} H_{\mu\alpha\beta}(f_\nu) + O(\theta^2)\ . \tag{27}$$

Now, introducing (13) and (27) into (10), integrating by parts and using (24), we arrive at our key result

$$\begin{aligned} I_{\mathrm{M}} = \int \mathrm{d}^3x \Bigg[&-\mu f^\mu\left(f_\mu + 2\theta^{\rho\beta} b_{\mu\rho\beta}\right) \\ &+ \frac{\kappa}{2}\,\epsilon^{\alpha\mu\nu}\left(f_\alpha + 2\theta^{\rho\beta} b_{\alpha\rho\beta}\right)\partial_\mu f_\nu \\ &+ \frac{\kappa}{6}\,\epsilon^{\alpha\mu\nu}\theta^{\rho\beta} f_\alpha \partial_\rho f_\mu \partial_\beta f_\nu\Bigg] + O(\theta^2)\ , \end{aligned} \tag{28}$$

which, as can be verified using (1) and (13), corresponds precisely to the expansion, to the first non-trivial order in θ, of (6). In this way, we have shown, at order $\mathrm{O}(\theta)$, the equivalence between the theories described by the actions (6) and (7), and used this result, together with the ones in [15], to remove the obstacles arising in the generalization to the NC space of the bosonization in three dimensions, along the lines of [6]. All calculations have been performed using the traditional master action approach, and considering only the first non-trivial order in the NC parameter. As emphasized before, this is not a very restrictive imposition, as most results computed until today involve corrections of order $O(\theta)$ over ordinary space.

3 Second order

Now we analyze how our results extend to order $O(\theta^2)$. For the sake of simplicity, we consider, instead of (13), the natural choice

$$\hat{f}_\mu = f_\mu \ . \tag{29}$$

We then note from (27) that $\hat{A}_\mu$ will be of the form

$$\hat{A}_\mu = \frac{1}{2}\, f_\mu + \theta^{\alpha\beta} H_{\mu\alpha\beta}(f_\nu) + \theta^{\alpha\beta}\theta^{\rho\sigma} W_{\mu\alpha\beta\rho\sigma}(f_\nu) + O(\theta^3) \ , \tag{30}$$

where $H_{\mu\alpha\beta}(f_\nu)$ satisfies (24). In principle, $W_{\mu\alpha\beta\rho\sigma}(f_\nu)$ should be computed by expanding (12) to order $O(\theta^2)$ and then solving it. In order to do this, we should include the SWM to order $O(\theta^2)$ in our calculations. However, nothing of this will be necessary, because an interesting result that we will show is that, in fact, $W_{\mu\alpha\beta\rho\sigma}(f_\nu)$ will not contribute to our final result.

Now using (24), (29) and (30) we find

$$\epsilon^{\alpha\mu\nu}\hat{f}_\alpha\partial_\mu\hat{A}_\nu = \frac{1}{2}\,\epsilon^{\alpha\mu\nu} f_\alpha \times \left(\partial_\mu f_\nu + \frac{1}{4}\,\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu + 2\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\mu W_{\nu\rho\beta\sigma\varphi}(f)\right) + O(\theta^3) \ , \tag{31}$$

$$\epsilon^{\alpha\mu\nu}\hat{A}_\alpha\partial_\mu\hat{A}_\nu = \frac{1}{4}\,\epsilon^{\alpha\mu\nu} f_\alpha \times \left(\partial_\mu f_\nu + \frac{1}{2}\,\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu + 4\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\mu W_{\nu\rho\beta\sigma\varphi}(f) - \frac{1}{16}\,\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\rho f_\mu \partial_\varphi f_\nu \partial_\beta f_\sigma\right) + O(\theta^3) + \dots \ , \tag{32}$$

$$\epsilon^{\alpha\mu\nu}\hat{f}_\alpha\left(\hat{A}_\mu \star \hat{A}_\nu\right) = \frac{\mathrm{i}}{8}\,\epsilon^{\alpha\mu\nu} f_\alpha \left(\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu - \frac{1}{2}\,\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\rho f_\mu \partial_\varphi f_\nu \partial_\beta f_\sigma\right) + O(\theta^3) \ , \tag{33}$$

$$\epsilon^{\alpha\mu\nu}\hat{A}_\alpha\left(\hat{A}_\mu \star \hat{A}_\nu\right) = \frac{\mathrm{i}}{16}\,\epsilon^{\alpha\mu\nu} f_\alpha \left(\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu - \frac{3}{4}\,\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\rho f_\mu \partial_\varphi f_\nu \partial_\beta f_\sigma\right) + O(\theta^3) + \dots \ , \tag{34}$$

where the dots stand for total derivatives. Introducing (31)–(34) into (10) and the terms containing $W_{\nu\rho\beta\sigma\varphi}(f)$ remarkably cancel out, and we get

$$I_{\mathrm{M}} = \int \mathrm{d}^3x\, f_\alpha \times \left[-\mu f^\alpha + \frac{\kappa}{2}\,\epsilon^{\alpha\mu\nu}\left(\partial_\mu f_\nu + \frac{1}{3}\,\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu - \frac{1}{16}\,\theta^{\rho\beta}\theta^{\sigma\varphi}\partial_\rho f_\mu \partial_\beta f_\sigma \partial_\varphi f_\nu\right)\right] + O(\theta^3) \ . \tag{35}$$

On the other hand, using (29), the expansion to order $O(\theta^2)$ of (6) is given by

$$I_B = \int \mathrm{d}^3x\, f_\alpha \times \left[-\mu f^\alpha + \frac{\kappa}{2}\,\epsilon^{\alpha\mu\nu}\left(\partial_\mu f_\nu + \frac{1}{3}\,\theta^{\rho\beta}\partial_\rho f_\mu \partial_\beta f_\nu\right)\right] + O(\theta^3) \ , \tag{36}$$

so that

$$I_{\mathrm{M}} = I_B - \frac{\kappa}{32}\epsilon^{\alpha\mu\nu}\theta^{\rho\beta}\theta^{\sigma\varphi}\int \mathrm{d}^3x\, f_\alpha \partial_\rho f_\mu \partial_\beta f_\sigma \partial_\varphi f_\nu + O(\theta^3) \ . \tag{37}$$

In this way, we have shown that, as anticipated, the "non-Abelian-like" nature of NC theories finally prevails at order $O(\theta^2)$, as the duality between (6) and (7) is lost, due to the f-quartic term which has finally appeared at this order. However, a remarkable thing to notice is that the theory (35) is local, unlike what happens to the non-Abelian, but commutative case [21]. Notice, also, that the Abelian nature of the theory allowed to compute the explicit expression of the f-quartic term, which is something that could not be done in the non-Abelian case (see (9)).

We should also contrast our results with the other previous ones in [16]. Notice that, according to our calculations, the dual of (7) is given by (35), which differs from (17) of [16], and involves a much simpler expression. The remarkably simple form of (35) followed from the key observation that $W_{\mu\alpha\beta\rho\sigma}(f_\nu)$ in (30) does not contribute to the final result.

In this work, we have considered calculations at orders $O(\theta)$ and $O(\theta^2)$. In principle, this perturbative approach could be extended to higher orders in θ. Notice that, at order $O(\theta^3)$, the term $W_{\mu\alpha\beta\rho\sigma}(f_\nu)$ in (30) will finally contribute, and to compute its explicit form would involve to consider the correction at order $O(\theta^2)$ over the SWM (2). In principle, this poses no problem other that the increasing algebraic difficulties. However, it can be verified that, in solving (12) at order $O(\theta^3)$, new difficulties arise, which are similar to the ones already known from the non-Abelian, but commutative case, suggesting that, at order $O(\theta^3)$, local solutions no longer exist. However, something that we were able to verify is that the $O(\theta^3)$ correction over (35) is of order $O(f^5)$. This suggests that, in general, higher $O(\theta^n)$ corrections over (35) should be of order $O(f^{(n+2)})$.

To finish, we conclude by pointing out that three-dimensional bosonization in NC space should be possible only when NC effects are weak, and only the first non-trivial corrections over ordinary space, that is to say $O(\theta)$ corrections, are relevant. This is not a very restrictive imposition, as most results till date in NC spaces involve only such kind of corrections.

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